FLOW EQUATIONS FOR THE RELEVANT PART
OF THE PURE YANG–MILLS ACTION

Ulrich ELLWANGER[1]

Laboratoire de Physique Théorique et Hautes Energies[2]

Université de Paris XI, bâtiment 211, 91405 Orsay Cedex, France

Manfred HIRSCH[3] and **Axel WEBER**[4]

Institut für Theoretische Physik

Universität Heidelberg, Philosophenweg 16, D–69120 Heidelberg, FRG

and

Laboratoire de Physique Théorique et Hautes Energies[2]

Université de Paris XI, bâtiment 211, 91405 Orsay Cedex, France

Abstract

Wilson's exact renormalization group equations are derived and integrated for the relevant part of the pure Yang–Mills action. We discuss in detail how modified Slavnov–Taylor identities controle the breaking of BRST invariance in the presence of a finite infrared cutoff k through relations among different parameters in the effective action. In particular they imply a nonvanishing gluon mass term for nonvanishing k. The requirement of consistency between the renormalization group flow and the modified Slavnov–Taylor identities allows to controle the self–consistency of truncations of the effective action.

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[1] e–mail: ellwange@qcd.th.u–psud.fr

[2] Laboratoire associé au Centre National de la Recherche Scientifique – URA 63

[3] Supported by a DAAD–fellowship HSP II financed by the German Federal Ministry for Research and Technology; e–mail: hirsch@qcd.th.u–psud.fr

[4] Supported by a LGFG–fellowship of the Land Baden–Württemberg and a DAAD–fellowship; e–mail: weber@qcd.th.u–psud.fr

1   Introduction

One of the most important and most interesting problems in quantum field theory today is the description of non–perturbative phenomena. The main tools used so far to tackle these problems are Schwinger–Dyson equations and the formulation on a (finite) space–time lattice. The difficulties with both methods can be traced back to the fact that in quantum field theory effects on all length scales are involved in an essential way.

The last years have witnessed the emergence of a new technique which may be capable of surmounting these difficulties. It originated much earlier in the work of Wilson et al. [1], and is based on the idea that the effects of different length scales should be taken into account one after the other rather than all at a time. The resulting 'exact renormalization group equations' or 'flow equations' are formulated directly in continuous Euclidean space–time, thereby avoiding any lattice artifacts. Polchinski [2] was the first to present a version adapted to the needs of quantum field theory. His exact renormalization group equation determines the flow of the effective Lagrangian; by taking the Legendre transform a flow equation for the effective action has been obtained later [3]–[5]. The effective action being the generating functional of one–particle irreducible Green functions, this formulation is more suited to the applications we have in mind than the older one involving the effective Lagrangian, which generates the truncated connected Green functions.

Technically, the flow equation for the effective action is obtained by introducing an infrared momentum cutoff k which is then varied continuously. At a large scale $\bar{k}$ only the quantum fluctuations on scales above $\bar{k}$ are integrated out, so that the corresponding effective action $\Gamma_{\bar{k}}$ incorporates the microscopic physics in the sense of an effective field theory. In the limit $k \to 0$ the effective action becomes the usual one, generating the physical one–particle irreducible Green functions. The flow equation now provides a differential equation interpolating between $k = \bar{k}$ and $k = 0$, so given $\Gamma_{\bar{k}}$ one can in principle calculate the physical effective action by just integrating the flow equation. Of course in practice it will not be possible to perform the integration exactly, since the functional Γ_k will have to be described in general by infinitely many k–dependent parameters. What renders this approach useful, then, is the existence of approximations which take into account only a finite number of parameters and provide nevertheless a sensible description of the relevant physical effects.

Historically, flow equations have been used first to simplify proofs of perturbative renormalizability [2],[4],[6]–[9]. Later they have been employed for the description of non–perturbative phenomena like phase transitions, critical exponents, bound states and condensates [5, 11]. Of course it is very desirable to apply this method to gauge theories like QCD. However one encounters here the obvious problem that the infrared cutoff k breaks the gauge or BRST invariance. Recently two different ways to cope with this problem have been proposed. In [12] background fields have been introduced in order to maintain gauge invariance. However the resulting additional dependence of the effective action Γ_k on the background field leads to new difficulties. In [8]–[10] it has been suggested to use modified Ward or Slavnov–Taylor identities to treat the symmetry breaking induced by the infrared cutoff. These identities guarantee the gauge or BRST invariance of the physical effective

action at $k = 0$. We follow here the formulation presented in [10].

The aim of the present paper is to demonstrate the suitability of this approach to calculations in gauge theories. For definiteness we consider SU(3)-Yang-Mills theory, but the concept is by no means limited to this particular case. The emphasis here is on the method, so we shall work with a rather simple but nevertheless non–trivial approximation for the k–dependent effective action. However, the formalism is general and can be combined with other techniques already developed in connection with the flow equations.

The paper is organized as follows: In section 2 we present the flow equations and modified Slavnov–Taylor identities following [10], but using a more concise and generalizable notation. In section 3 we outline how this framework can be employed to calculate the physical effective action, and we present the approximation we shall use in the following. Section 4 represents the main part of the paper and contains the results from the integration of the flow equations. We shall find in particular that the BRST invariance of the original theory provides us with the means to control the approximation we have introduced. Section 5 contains our conclusions and a short outlook.

2 Flow equations and modified Slavnov–Taylor identities

Throughout this paper we consider SU(3)–Yang–Mills theory in four–dimensional Euclidean space–time, although the extension to general SU(N)–Yang–Mills theories in d–dimensional space–time does not involve additional complications.

We take the classical action to be

$$
\begin{aligned}
S \;=\; & \int \mathrm{d}^4 x \left\{ \tfrac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{1}{2\alpha} \partial_\mu A_\mu^a \partial_\nu A_\nu^a + \partial_\mu \bar{c}^a \left(\partial_\mu c^a + g f^a{}_{bc} A_\mu^b c^c \right) \right. \\
& \left. - K_\mu^a \left(\partial_\mu c^a + g f^a{}_{bc} A_\mu^b c^c \right) - L^a \tfrac{1}{2} g f^a{}_{bc} c^b c^c + \bar{L}^a \frac{1}{\alpha} \partial_\mu A_\mu^a \right\}
\end{aligned}
\tag{1}
$$

with

$$
F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^a{}_{bc} A_\mu^b A_\nu^c \,,
\tag{2}
$$

where we have included the usual gauge fixing and ghost parts and coupled external sources to the BRST variations

$$
\begin{aligned}
\delta A_\mu^a \;&=\; \left(\partial_\mu c^a + g f^a{}_{bc} A_\mu^b c^c \right) \zeta \\
\delta c^a \;&=\; \tfrac{1}{2} g f^a{}_{bc} c^b c^c \zeta \\
\delta \bar{c}^a \;&=\; -\frac{1}{\alpha} \partial_\mu A_\mu^a \zeta \,.
\end{aligned}
\tag{3}
$$

Here ζ is a Grassmann parameter. The invariance of S at $\bar{L} = 0$ under BRST transformations can then be expressed as

$$
0 \;=\; \int \mathrm{d}^4 x \left\{ \delta A_\mu^a \frac{\delta S}{\delta A_\mu^a} + \delta c^a \frac{\delta S}{\delta c^a} + \delta \bar{c}^a \frac{\delta S}{\delta \bar{c}^a} \right\} \bigg|_{\bar{L}=0}
$$

$$= \zeta \int \mathrm{d}^4 x \left\{ \frac{\delta S}{\delta K_\mu^a} \frac{\delta S}{\delta A_\mu^a} - \frac{\delta S}{\delta L^a} \frac{\delta S}{\delta c^a} - \frac{\delta S}{\delta \bar{L}^a} \frac{\delta S}{\delta \bar{c}^a} \right\} \bigg|_{\bar{L}=0} . \tag{4}$$

To derive the flow equations we add an infrared momentum cutoff term

$$\Delta S_k = \tfrac{1}{2} A \cdot R_k \cdot A + \bar{c} \cdot \tilde{R}_k \cdot c \tag{5}$$

to the action S. Here we have introduced a matrix notation which will be useful in the following. It is most easily visualized in momentum space, where the matrix product implies integrations over momenta as well as summations over group and Lorentz indices. As an example, $A \cdot R_k \cdot A$ actually stands for

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} A_\mu^a(-p) R_{k,\mu\nu}^{ab}(p,-q) A_\nu^b(q) . \tag{6}$$

Imposing energy–momentum conservation and global SU(3)–invariance we have

$$R_{k,\mu\nu}^{ab}(p,-q) = R_{k,\mu\nu}(p)\delta^{ab}(2\pi)^4\delta(p-q)$$
$$\tilde{R}_k^{ab}(p,-q) = \tilde{R}_k(p)\delta^{ab}(2\pi)^4\delta(p-q) . \tag{7}$$

The functions $R_{k,\mu\nu}(p)$ and $\tilde{R}_k(p)$ introduce smooth infrared cutoffs k in the gluon and ghost field propagators and vanish identically at $k = 0$,

$$R_{k,\mu\nu}(p), \ \tilde{R}_k(p) \to 0 \quad \text{as} \quad k \to 0 . \tag{8}$$

Furthermore,

$$R_{k,\mu\nu}(p), \ \tilde{R}_k(p) \to 0 \quad \text{`rapidly' as} \quad p^2 \to \infty , \tag{9}$$

so that R_k and $\tilde{R}_k$ act simultaneously as ultraviolet cutoffs in the integrals on the r.h.s. of (16) and (21) below. Explicit expressions for R_k and $\tilde{R}_k$ will be given later.

The k–dependent generating functional of connected Green functions W_k is defined by

$$e^{W_k[J,\chi,\bar{\chi},K,L,\bar{L}]} = \int \mathcal{D}_{reg}(A,c,\bar{c}) e^{-\Delta S_k - S + J \cdot A + \bar{\chi} \cdot c + \bar{c} \cdot \chi} , \tag{10}$$

where we have attached the index "reg" to the functional integration measure to indicate that the functional integral has been regularized in the ultraviolet. This regularization is necessary to give a meaning to (10), and it should be independent of k and preserve the symmetries of S. It is not quite clear whether such a regularisation exists in a non–perturbative sense. In the following we shall nevertheless assume its existence in order to present a simple derivation of the flow equations and the modified Slavnov–Taylor identities (STI) and to illustrate the physical meaning of the quantities under consideration. Once we have obtained the flow equations and modified STI, however, we can use them without reference to (10) and thus do not rely on the existence of an appropriate regularization any more, as has been emphasized in [10]. In a more rigorous sense, the theory may directly be defined via the flow equations and modified STI [8].

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So let us, for the moment, use (10) in order to define the functional W_k. Since ΔS_k vanishes for $k = 0$, we see from (10) that W_0 is the generating functional of the physical connected Green functions.

For general k we pass over to the Legendre transform with respect to the sources $J, \chi, \bar{\chi}$,

$$\Gamma_k[A, c, \bar{c}, K, L, \bar{L}] = A \cdot J + \bar{\chi} \cdot c + \bar{c} \cdot \chi - W_k[J, \chi, \bar{\chi}, K, L, \bar{L}] , \tag{11}$$

$$\text{where} \quad A = \frac{\delta W_k}{\delta J} , \quad c = \frac{\delta W_k}{\delta \bar{\chi}} , \quad \bar{c} = -\frac{\delta W_k}{\delta \chi} . \tag{12}$$

Finally, $\hat{\Gamma}_k$ is defined by

$$\Gamma_k = \hat{\Gamma}_k + \Delta S_k . \tag{13}$$

At $k = 0$, $\hat{\Gamma}_0 = \Gamma_0$ becomes the physical effective action or the generating functional of one–particle irreducible Green functions.

Differentiating $\hat{\Gamma}_k$ with respect to k leads to the flow equations. To cast them into a concise and generalizable form, we introduce the superfields

$$\Phi = (c, \bar{c}, A) \quad \text{and} \quad \bar{\Phi} = (-\bar{c}, c, A) \tag{14}$$

and (for later use) the external supersources

$$Q = (L, \bar{L}, K) . \tag{15}$$

The flow equations finally read [3]–[5]

$$\partial_k \hat{\Gamma}_k = \tfrac{1}{2} \text{STr} \left[\partial_k \mathcal{R}_k \cdot \left(\frac{\delta^2 \hat{\Gamma}_k}{\delta \bar{\Phi} \delta \Phi} + \mathcal{R}_k \right)^{-1} \right] . \tag{16}$$

Here the inverse is to be taken in the matrix sense including the momenta as indices. The elements of the matrix of second derivatives are in our notation given by

$$\left(\frac{\delta^2 \hat{\Gamma}_k}{\delta \bar{\Phi} \delta \Phi} \right)_{ij} (p, q) = (2\pi)^8 \frac{\delta^2 \hat{\Gamma}_k}{\delta \bar{\Phi}_i(-p) \delta \Phi_j(-q)} , \tag{17}$$

where i and j denote the group and (in the case of gluon fields) the Lorentz indices. $\mathcal{R}_k$ stands for the matrix of cutoff terms,

$$\mathcal{R}_k = \begin{pmatrix} \tilde{R}_k & 0 & 0 \\ 0 & \tilde{R}_k & 0 \\ 0 & 0 & R_k \end{pmatrix} , \tag{18}$$

written in a block matrix notation. STr denotes the supertrace,

$$\text{STr}(A) = \text{Tr}(M \cdot A) \tag{19}$$

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for some (super)matrix A, where

$$M = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{20}$$

in the above block matrix notation. Again, the trace includes summation over the momenta.

Flow equations for the one–particle irreducible n–point functions can be derived from (16) by taking derivatives with respect to the gluon and ghost fields as well as to the external sources K, L and $\bar{L}$, setting them equal to zero after differentiation as usual. The complete set of flow equations is then represented by an infinite system of coupled differential equations for the n–point functions. It is possible to visualize them in the form of Feynman diagrams. The k–derivative of a n–point function is given by a finite sum of one–loop graphs, where in contradistinction to the usual diagrams of perturbation theory the propagators and vertices are dressed. All the diagrams are infrared and ultraviolet finite provided the cutoff functions (7) appearing in (16) are appropriately chosen. We emphasize that despite their one–loop appearance the equations are still exact.

The introduction of the cutoff term ΔS_k in the action breaks the BRST invariance for $\bar{L} = 0$ explicitly. Using the invariance of the functional integration measure the symmetry breaking can be quantified and expressed in the form of modified STI for $\hat{\Gamma}_k$ [10]:

$$\frac{\delta \hat{\Gamma}_k}{\delta K} \cdot \frac{\delta \hat{\Gamma}_k}{\delta A} - \frac{\delta \hat{\Gamma}_k}{\delta L} \cdot \frac{\delta \hat{\Gamma}_k}{\delta c} - \frac{\delta \hat{\Gamma}_k}{\delta \bar{L}} \cdot \frac{\delta \hat{\Gamma}_k}{\delta \bar{c}} \bigg|_{\bar{L}=0} = \mathrm{STr} \left[\mathcal{R}_k \cdot \frac{\delta^2 \hat{\Gamma}_k}{\delta Q \delta \Phi} \cdot \left(\frac{\delta^2 \hat{\Gamma}_k}{\delta \bar{\Phi} \delta \Phi} + \mathcal{R}_k \right)^{-1} \right] \bigg|_{\bar{L}=0} . \tag{21}$$

The external supersources Q have been introduced in (15). In (21) the left hand side represents the usual variation under BRST transformations, while the right hand side specifies the symmetry breaking term, vanishing at $k = 0$. Both sides are to be taken at $\bar{L} = 0$.[5]

The modified STI (21) are similar in form to the flow equations (16). Expanded in the fields and external sources the former are also equivalent to an infinite system of coupled nonlinear equations for the k–dependent one–particle irreducible n–point functions. Again, the trace term can be represented as a finite sum of one–loop diagrams with dressed vertices and propagators. Infrared and ultraviolet finiteness is guaranteed by appropriate choices for the cutoff functions.

Another useful identity is [8]–[10]

$$\partial \cdot \frac{\delta \hat{\Gamma}_k}{\delta K} = \frac{\delta \hat{\Gamma}_k}{\delta \bar{c}} , \tag{22}$$

where ∂ stands for differentiation with respect to the space–time variable. (22) is obtained easily by performing a redefinition of the field $\bar{c}$ in (10).

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[5]A different but equivalent formulation is presented in [8, 9]. The modified STI are called "fine-tuning conditions" there.

The $\bar{L}$–dependence of $\hat{\Gamma}_k$ is trivially fixed to be

$$\frac{\delta \hat{\Gamma}_k}{\delta \bar{L}} = \frac{1}{\alpha} \partial \cdot A \,. \tag{23}$$

The last two identities are already valid at the classical level, i.e. with $\hat{\Gamma}_k$ replaced by S. (22) and (23) then simply state the absence of quantum corrections to the classical relations when expressed in terms of $\hat{\Gamma}_k$.

The crucial point, as shown in [10], is that the flow equations respect the modified STI, as well as (22) and (23), in the following sense: Once the latter are satisfied at a certain scale $k = \bar{k}$, they are also satisfied for all $k < \bar{k}$ provided $\hat{\Gamma}_k$ is obtained from $\hat{\Gamma}_{\bar{k}}$ by integration of the flow equations. In particular, at $k = 0$ the physical effective action $\hat{\Gamma}_0$ fulfills the usual STI. Hence the meaning of the modified STI for $\hat{\Gamma}_k$ is that they imply the BRST invariance of the physical effective action at $k = 0$.

We remark that the modified STI, as well as the flow equations, can be extended trivially to include matter fields.

3   A simple approximation

The physical quantity one is interested in is the effective action $\hat{\Gamma}_0$. Let us briefly outline how the flow equations together with the modified STI can be used to calculate $\hat{\Gamma}_0$ in principle. The general idea is to begin with a certain $\hat{\Gamma}_{\bar{k}}$ at a scale $\bar{k}$ which has to be large in comparison with all physical scales of the theory, and then to integrate the flow equations down to $k = 0$. Although the exact form of $\hat{\Gamma}_{\bar{k}}$ is not known a priori, we expect from universality that the resulting $\hat{\Gamma}_0$ is widely independent of the starting point $\hat{\Gamma}_{\bar{k}}$. In the present case of an asymptotically free theory the relevant parameters of $\hat{\Gamma}_{\bar{k}}$ are the coefficients with non–negative mass dimension in an expansion of $\hat{\Gamma}_{\bar{k}}$ in fields and momenta.[6] In addition it is essential for $\hat{\Gamma}_{\bar{k}}$ to possess the right symmetry properties. In our case of BRST invariance this is accomplished by imposing the modified STI.

The main problem in practical calculations is of course the integration of the flow equations. Since they represent an infinite system of coupled differential equations for the n–point functions, an exact solution seems impossible. Instead one is forced to truncate the system by taking into account only a finite number of parameters representing the k–dependent effective action $\hat{\Gamma}_k$. In practice one then inserts the truncated representation of $\hat{\Gamma}_k$ in the r.h.s. of (16) and projects out the contributions to the flow of the parameters under consideration on the l.h.s. Similarly the modified STI represent an infinite system of coupled nonlinear equations for the n–point functions. By using the truncated form of $\hat{\Gamma}_k$ in (21) and performing appropriate projections one arrives at a set of equations for the parameters representing $\hat{\Gamma}_k$ at every scale k.

Now taking a functional $\hat{\Gamma}_{\bar{k}}$ which fulfills the modified STI exactly and integrating the flow equations down to a lower scale k exactly, we would obtain some $\hat{\Gamma}_k$ which again

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[6]We include here the parameters which are usually termed marginal.

satisfies the modified STI. However, due to the necessary approximations described above this is no longer true in practice. As a consequence we cannot expect the physical effective action $\hat{\Gamma}_0$ to be BRST–invariant if we just integrate the truncated flow equations.

Let us look at this problem more closely in the special case of the gluon mass. As will be shown below, for a pure SU(3)–Yang–Mills theory the modified STI demand a gluon mass term in $\hat{\Gamma}_k$ as long as $k \neq 0$. Stated differently, for the physical effective action at $k = 0$ to be BRST–invariant, the effective action at a scale $k \neq 0$ must contain a precisely specified gluon mass term. Due to the truncations, however, the gluon mass will deviate from the value prescribed by the modified STI in the course of the integration of the flow equations, even if one started with the correct value at a higher scale. Since the mass is a relevant parameter, one will thus not end up with a BRST–invariant effective action at $k = 0$.

These problems are solved if we impose the truncated form of the modified STI on $\hat{\Gamma}_k$ at *every* scale k. In this way we are 'near' to the $\hat{\Gamma}_k$ satisfying the exact modified STI during the whole flow and our approximation to the physical effective action $\hat{\Gamma}_0$ is as close as possible to exact BRST invariance within our restricted space of parameters. Since in an exact calculation the modified STI would be fulfilled automatically given the correct starting point, we can in fact turn the argument around and consider the deviations due to the truncations as providing an internal validity check for the approximation.

To be more precise, we have to divide our set of parameters into 'dependent' and 'independent' ones, where the dependent parameters are entirely fixed by the modified STI in terms of the independent ones. The number of independent parameters is the same as in perturbation theory with BRST–invariant regularization. Now we use the truncated flow equations to determine the flow of the independent parameters, but rely on the modified STI to obtain the flow of the dependent ones. On the other hand, the flow of the dependent parameters could also be read off directly from the flow equations without recourse to the modified STI. Due to the truncations, the two results obtained in this way will in general differ slightly. This difference can in turn be regarded as a measure of the validity of the approximation.

After this general discussion let us now return to the special case of a SU(3)–Yang–Mills theory. The classical action including external source terms was given in (1). A natural ansatz for a truncated k–dependent effective action would be to include all perturbatively relevant couplings, i.e. all couplings with non–negative mass dimension. So we choose as a truncation

$$
\begin{aligned}
\hat{\Gamma}_k &= \int \mathrm{d}^4x \left\{ \frac{Z_k^A}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{Z_k^A}{2\alpha_k} \partial_\mu A_\mu^a \partial_\nu A_\nu^a + \frac{Z_k^A}{2} m_k^2 A_\mu^a A_\mu^a \right. \\
&\quad + Z_k^c \partial_\mu \bar{c}^a \left(\partial_\mu c^a + (Z_k^A)^{1/2} g_k^K f^a{}_{bc} A_\mu^b c^c \right) - Z_k^c K_\mu^a \left(\partial_\mu c^a + (Z_k^A)^{1/2} g_k^K f^a{}_{bc} A_\mu^b c^c \right) \\
&\quad \left. - \frac{Z_k^c (Z_k^A)^{1/2}}{2} g_k^L L^a f^a{}_{bc} c^b c^c + \frac{1}{\alpha^{(0)}} \bar{L}^a \partial_\mu A_\mu^a \right\} \,,
\end{aligned}
\tag{24}
$$

where

$$
F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + (Z_k^A)^{1/2} g_k^A f^a{}_{bc} A_\mu^b A_\nu^c \,.
\tag{25}
$$

A priori one would expect the coefficients of all the terms in (24) to be independent quantities. However, some of them are related, as will be explained in detail below. For similar reasons certain terms are omitted which would be present from dimensional reasoning alone. Furthermore, powers of the wave function renormalization constants Z_k^A and Z_k^c for the gluon and ghost fields have been extracted from the coupling constants, which will prove to be convenient in the course of the calculations.

Let us first consider the coefficient of the $\bar{L}A$–vertex, which is fixed to be the classical one by (23). For clarity we denote the 'classical' gauge fixing constant α by $\alpha^{(0)}$ in the following. It appears in the BRST transformation (3) and through the $\bar{L}A$–vertex enters in the modified STI. Note that the gauge fixing constant in $\hat{\Gamma}_k$, i.e. the coefficient of the longitudinal part of the gluon two–point function, will in general be k–dependent, whereas $\alpha^{(0)}$ remains fixed during the flow.

The identity (22) demands that $\hat{\Gamma}_k$ depend only on the combination $\partial\bar{c} - K$, which implies the equality of the corresponding couplings. In particular the formally relevant coefficients of the $\bar{c}cAA$– and the $\bar{c}c\bar{c}c$–vertex are related to the irrelevant $KcAA$– and $KcKc$–couplings respectively and are thereby also rendered irrelevant. Hence they are not included in our ansatz (24). The same property of $\hat{\Gamma}_k$ prevents the appearance of a mass term for the ghost field.[7]

We have introduced a k–dependent gauge fixing constant α_k and a gluon mass m_k. The coefficients of the AAA–, the KcA– and the Lcc–vertex are in general different and have been denoted by the three coupling constants g_k^A, g_k^K and g_k^L. We would like to remark that due to the relations between the different couplings implied by the modified STI the number of independent couplings is equal to the number of independent couplings in the relevant part of a BRST–invariant action. In particular we shall show in the next section how the modified STI fix the couplings m_k^2, α_k, g_k^K and g_k^L in terms of g_k^A and the product $Z_k^A\alpha^{(0)}$ for every scale k.

As implied by (25), the coupling constant of the four–gluon vertex is given by $(g_k^A)^2$. This is a further approximation which is not justified by the modified STI. In fact these predict a deviation of the four–gluon coupling constant from the value $(g_k^A)^2$. However we shall see in the next section that within the region of validity of our approximation the difference between the coupling constants g_k^A, g_k^K and g_k^L may be neglected for calculational purposes. We therefore expect the same to hold true for the deviation of the four–gluon coupling constant from $(g_k^A)^2$.

Furthermore the modified STI enforce the appearance of an additional four–gluon coupling with the group–tensorial structure

$$\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} . \tag{26}$$

The same structure is also generated by the flow equations, as it of course must be for consistency reasons. We neglect this coupling in our approximation because it does not contribute directly to the flow of the coupling constants g_k^A, g_k^K and g_k^L, which are our primary objects of interest. In fact it only contributes to the flow of the gluon mass and

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[7]Observe however that the cutoff function (28) does introduce a mass–like term $\sim k^2$ for the ghost field.

the four–gluon coupling itself. Concerning these additional approximations one should, however, always bear in mind that the comparison of the flow of the dependent parameters taken directly from the flow equations with the one determined via the modified STI provides us with a measure of the validity of the used truncation, thus indicating every eventual breakdown of the approximation. This concludes the discussion of the ansatz for $\hat{\Gamma}_k$ used in the following.

To complete the description of the set–up used for the calculations in the next section we need to specify the form of the cutoff functions. These functions may, and in our case will, contain the k–dependent parameters appearing in $\hat{\Gamma}_k$. We choose

$$
R_{k,\mu\nu}(p) = Z_k^A \left\{ \frac{(p^2 + m_k^2)e^{-(p^2+m_k^2)/k^2}}{1 - e^{-(p^2+m_k^2)/k^2}} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \right.
$$
$$
\left. + \frac{(p^2/\alpha_k + m_k^2)e^{-(p^2/\alpha_k+m_k^2)/k^2}}{1 - e^{-(p^2/\alpha_k+m_k^2)/k^2}} \frac{p_\mu p_\nu}{p^2} \right\} \tag{27}
$$

as the cutoff function for the gluons and

$$
\tilde{R}_k(p) = Z_k^c \frac{p^2 e^{-p^2/k^2}}{1 - e^{-p^2/k^2}} \tag{28}
$$

for the ghost field. They have the demanded properties, namely to introduce infrared cutoffs in the respective propagators and to fulfill the conditions (8) and (9), provided that $\alpha_k \geq 0$ and $\lim_{k\to 0} m_k^2 \geq 0$ (typically one finds $\lim_{k\to 0} m_k^2 = 0$, cf. section 4).

The choice (27) may appear unnecessarily complicated at first sight. However this particular form is chosen to give the propagator appearing in the loop integrals generated by the trace terms in (16) and (21) a fairly simple structure. Employing our ansatz for $\hat{\Gamma}_k$ we obtain for the gluon propagator

$$
\frac{1}{Z_k^A} \left\{ \frac{1 - e^{-(p^2+m_k^2)/k^2}}{p^2 + m_k^2} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) + \frac{1 - e^{-(p^2/\alpha_k+m_k^2)/k^2}}{p^2/\alpha_k + m_k^2} \frac{p_\mu p_\nu}{p^2} \right\} \delta^{ab} . \tag{29}
$$

A similar calculation for the ghost propagator yields

$$
\frac{1}{Z_k^c} \frac{1 - e^{-p^2/k^2}}{p^2} \delta^{ab} . \tag{30}
$$

Due to the particular structure of the propagators we are able to perform all loop integrals appearing in our approximation analytically. However, elegance often comes at a price. In our case the drawback is that the cutoff terms involve the k–dependent parameters Z_k^A, Z_k^c, m_k^2 and α_k. Hence on the r.h.s. of the flow equations the k–derivatives of all these parameters appear on evaluating $\partial_k \mathcal{R}_k$, as will be seen explicitly in the next section.

Note that the functions appearing in (28) and (30) as well as the coefficients of the transverse and longitudinal momentum projectors in (27) and (29) are analytical functions of p^2, even if $m_k^2 < 0$. In particular they do not contain poles. The limits $p^2 \to 0$ and $m_k^2 \to 0$ in (27) are, despite appearance, well–defined and interchangeable. This guarantees

that the functional $\Gamma_k = \hat{\Gamma}_k + \Delta S_k$ is local for all values of m_k, i.e. the n–point functions can be expanded in powers of the momenta, which should be the case for a sensible infrared cutoff.

One might be tempted to choose different mass parameters m_k in the transverse and the longitudinal parts in (24) and (27) and hence also in (29), since it is only the longitudinal part which will be fixed by the modified STI. However, in this case the coefficient of the non–local term $p_\mu p_\nu / p^2$ in (24) and (27) would not vanish in the limit $p^2 \to 0$, and Γ_k would not be local.

4   Solving the truncated flow equations

We shall now turn to the integration of the flow equations using the concepts and the approximations explained in detail in the previous section. Employing the ansatz (24) the flow equations and the modified STI take on a characteristic form which we shall sketch in the following.

We expand the modified STI in the fields and external sources to obtain equations for the one–particle irreducible n–point functions. Expanding these in turn in powers of momenta and performing appropriate projections we arrive at the following set of equations:

$$
\mu_k \equiv \frac{m_k^2}{k^2} = \frac{3g_k^2}{(4\pi)^2} \left(f_1^\mu(\mu_k, \alpha_k) + \frac{\alpha_k}{Z_k^A \alpha^{(0)}} f_2^\mu(\mu_k, \alpha_k) \right)
$$

$$
\frac{\alpha_k}{Z_k^A \alpha^{(0)}} - 1 = \frac{3g_k^2}{(4\pi)^2} \left(f_1^\alpha(\mu_k, \alpha_k) + \frac{\alpha_k}{Z_k^A \alpha^{(0)}} f_2^\alpha(\mu_k, \alpha_k) \right)
$$

$$
\frac{g_k^A - g_k^K}{g_k^K} = \frac{3g_k^2}{(4\pi)^2} \left(f_1^A(\mu_k, \alpha_k) + \frac{\alpha_k}{Z_k^A \alpha^{(0)}} f_2^A(\mu_k, \alpha_k) \right)
$$

$$
\frac{g_k^L - g_k^K}{g_k^K} = \frac{3g_k^2}{(4\pi)^2} \left(f_1^L(\mu_k, \alpha_k) + \frac{\alpha_k}{Z_k^A \alpha^{(0)}} f_2^L(\mu_k, \alpha_k) \right) . \tag{31}
$$

Here we have introduced the dimensionless mass parameter $\mu_k = m_k^2/k^2$. The f_j^i are complicated functions of μ_k and α_k and do not depend explicitly on k. Hence k enters in the system explicitly only through the relation between m_k^2 and μ_k.

Furthermore we have made still another approximation by equating the three coupling constants g_k^A, g_k^K and g_k^L on the r.h.s. and denoting them collectively by g_k. Without this approximation we would have to replace the r.h.s. of (31) by a sum of similar expressions with g_k^2 replaced in turn by the products $(g_k^A)^2$, $g_k^A g_k^K$, ... of the three coupling constants. The factor of three in front of g_k^2 reflects the number of colors, while $1/(4\pi)^2$ originates from the integration measure of the loop integrals.

We see that in the perturbative limit of vanishing coupling constants $g_k \to 0$ the modified STI lead to the usual relations $m_k^2 = 0$, $Z_k^A/\alpha_k = 1/\alpha^{(0)}$ and $g_k^A = g_k^K = g_k^L$. The last two equations in (31) with the three coupling constants set equal to g_k on the r.h.s. can thus be considered as the first step in an iterational procedure to determine the values of g_k^A, g_k^K and g_k^L. What is more, these two equations make it easy to check whether the

approximation of equating the coupling constants is admissible. In fact we shall find this approximation to be valid in the region of parameter space we are interested in, which in turn is limited by the validity of the general approximation (24). Since we consider an asymptotically free theory and the starting scale $\bar{k}$ is supposed to be large as compared with the physical scales of the theory, e.g. the confinement scale, we have $g_{\bar{k}} \ll 1$ and the approximation of equal coupling constants is certainly acceptable for the starting point $\hat{\Gamma}_{\bar{k}}$.

In the same way we could have treated the four–gluon coupling constant as different from $(g_k^A)^2$ in the ansatz (24) and then relate it to $(g_k^A)^2$ through an equation similar to (31). However, we expect the difference between these quantities to be negligible within the region of validity of our approximation.

We observe that Z_k^A enters in (31) only through the combination $Z_k^A \alpha^{(0)}$, and Z_k^c does not appear at all. This simplification was in fact the reason for extracting powers of the wave function renormalization constants from the couplings in (24). By counting the parameters and the equations — regarding g_k^A, g_k^K and g_k^L as different for the moment — we find that the modified STI in the form (31) fix μ_k, α_k, g_k^K and g_k^L in terms of g_k^A and $Z_k^A \alpha^{(0)}$. Thus apart from the wave function renormalization constants we end up with two independent parameters, corresponding to g and α in perturbation theory with BRST-invariant regularization.

From now on we shall always use the approximation of equal coupling constants, i.e. $g_k = g_k^A = g_k^K = g_k^L$ unless otherwise stated, so we only need to consider the first two equations in (31). Given μ_k and α_k we obtain a system of quadratic equations for g_k^2 and $1/Z_k^A \alpha^{(0)}$, which can be solved easily. On the other hand we can calculate μ_k and α_k numerically as functions of g_k and $Z_k^A \alpha^{(0)}$.

Before we come to the discussion of the solutions of (31) let us consider the limit $\alpha^{(0)} \to 0$, i.e. the Landau gauge. One can show that in this limit the second equation in (31) implies $\alpha_k / Z_k^A \alpha^{(0)} = 1$ irrespective of the values of g_k and μ_k. Hence $\alpha_k = 0$ and the first equation in (31) becomes

$$\mu_k = \frac{3g_k^2}{(4\pi)^2} \left(f_1^\mu(\mu_k, 0) + f_2^\mu(\mu_k, 0) \right) , \tag{32}$$

which is much easier to work with. Also the functions f_j^i take on simpler, though still not enlightening forms in this limit.

We shall now turn to the discussion of the solutions of (31). It is easy to obtain solutions for small g_k iteratively. Neglecting terms of the order of g_k^4, the first two equations in (31) become

$$\mu_k = \frac{3g_k^2}{(4\pi)^2} \frac{3(\alpha_k - 1)}{8(\alpha_k + 1)}$$

$$\frac{1}{\alpha_k} - \frac{1}{Z_k^A \alpha^{(0)}} = \frac{3g_k^2}{(4\pi)^2} \frac{7 - 10\alpha_k - 5\alpha_k^2}{48(\alpha_k + 1)^2} . \tag{33}$$

For brevity we have used the quantity α_k on the r.h.s. However it can be replaced by $Z_k^A \alpha^{(0)}$ in the given order in g_k as follows from the second equation in (33).

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It appears problematic at first sight that the mass parameter μ_k becomes negative for $\alpha_k < 1$. However in the limit of vanishing momentum the gluonic two–point function obtained from Γ_k becomes

$$Z_k^A \frac{\mu_k k^2}{1 - e^{-\mu_k}} \delta_{\mu\nu} \,, \tag{34}$$

which is well–defined and non–negative for all values of μ_k. Since the propagator (29) is given by the inverse of this two–point function, the effective gluon mass is seen to be non–negative.

We shall refer to (33) as the one–loop STI in the following since they are valid up to second order in g_k. We remark that our expression for μ_k differs from the one–loop result obtained in [10], where a different cutoff function has been used. Hence the one–loop STI are non–universal in the sense that they depend on the choice of the cutoff terms.

The solutions of (31) for general values of g_k are shown in figure 1, where we have plotted μ_k as a function of the strong fine–structure constant $\alpha_S = g_k^2/4\pi$ for fixed values of α_k. The reason for using α_k as a parameter instead of $Z_k^A \alpha^{(0)}$ is just technical, but since both quantities are k–dependent, the use of $Z_k^A \alpha^{(0)}$ would not be more advantageous anyway. The values of α_k depicted in figure 1 are, from bottom to top, $\alpha_k = 0$, 1 and 3.

We shall see below that our approximation breaks down at $\alpha_S \approx 1.4$, so effects showing up at much greater values of α_S are considered as potential artifacts of the truncation. We finally remark that μ_k does not vanish for $g_k > 0$ in the 'Feynman gauge' $\alpha_k = 1$, although it does so in the one–loop approximation (33).

After the discussion of the modified STI let us now turn to the flow equations. Expanding (16) in the fields and external sources we obtain flow equations for the one–particle irreducible n–point functions. Some of these are represented diagrammatically in figure 2, where our ansatz for $\hat{\Gamma}_k$ has already been taken into account in that on the r.h.s. all vertices not appearing in (24) are omitted. Since the cutoff functions depend on k not only explicitly, but also implicitly through the parameters Z_k^c, Z_k^A, $m_k^2 = \mu_k k^2$ and α_k, we have

$$
\begin{aligned}
\partial_k R_{k,\mu\nu} &= \frac{\partial R_{k,\mu\nu}}{\partial k} + \frac{\partial R_{k,\mu\nu}}{\partial \ln Z_k^A} \partial_k \ln Z_k^A + \frac{\partial R_{k,\mu\nu}}{\partial \mu_k} \partial_k \mu_k + \frac{\partial R_{k,\mu\nu}}{\partial \alpha_k} \partial_k \alpha_k \\
\partial_k \tilde{R}_k &= \frac{\partial \tilde{R}_k}{\partial k} + \frac{\partial \tilde{R}_k}{\partial \ln Z_k^c} \partial_k \ln Z_k^c
\end{aligned}
\tag{35}
$$

on the r.h.s. of (16). Now we employ our ansatz for $\hat{\Gamma}_k$, expand the flow equations for the n–point functions in powers of external momenta and perform appropriate projections to arrive at equations for the parameters of $\hat{\Gamma}_k$. Using the approximation of equal coupling constants $g_k = g_k^A = g_k^K = g_k^L$ discussed above, the flow equations for the 'independent' parameters Z_k^A, Z_k^c and g_k finally take on the following form:

$$
\begin{aligned}
\frac{\partial \ln \lambda_k}{\partial \ln k^2} = \frac{3g_k^2}{(4\pi)^2} &\left(h_0^\lambda(\mu_k, \alpha_k) + h_A^\lambda(\mu_k, \alpha_k) \frac{\partial \ln Z_k^A}{\partial \ln k^2} + h_c^\lambda(\mu_k, \alpha_k) \frac{\partial \ln Z_k^c}{\partial \ln k^2} \right. \\
&\left. + h_\mu^\lambda(\mu_k, \alpha_k) \frac{\partial \mu_k}{\partial \ln k^2} + h_\alpha^\lambda(\mu_k, \alpha_k) \frac{\partial \alpha_k}{\partial \ln k^2} \right) \,,
\end{aligned}
\tag{36}
$$

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where λ_k represents the independent parameter. For definiteness we choose to take the equation for $\partial \ln g_k / \partial \ln k^2$ from the flow equation for the $\bar{c}cA$– or equivalently the KcA–vertex. The functions h_j^i appearing above depend only on the parameters μ_k and α_k and not explicitly on k. The terms involving $\partial \ln Z_k^A / \partial \ln k^2$ etc. originate from the expressions for $\partial_k R_{k,\mu\nu}$ and $\partial_k \tilde{R}_k$ in (35). Observe that the classical gauge fixing constant $\alpha^{(0)}$ does not enter in (36).

On the r.h.s. of (36) the k–derivatives of μ_k and α_k appear. Since these parameters are given in terms of g_k and $Z_k^A \alpha^{(0)}$ through the modified STI, one can obtain expressions for $\partial \mu_k / \partial \ln k^2$ and $\partial \alpha_k / \partial \ln k^2$ by taking the k–derivative of the first two equations in (31). We write them in the form

$$
\begin{aligned}
\frac{\partial \mu_k}{\partial \ln k^2} &= 2\mu_k \frac{\partial \ln g_k}{\partial \ln k^2} + \frac{3g_k^2}{(4\pi)^2} \left(h_A^\mu(\mu_k, \alpha_k, Z_k^A \alpha^{(0)}) \frac{\partial \ln Z_k^A}{\partial \ln k^2} \right. \\
&\quad \left. + h_\mu^\mu(\mu_k, \alpha_k, Z_k^A \alpha^{(0)}) \frac{\partial \mu_k}{\partial \ln k^2} + h_\alpha^\mu(\mu_k, \alpha_k, Z_k^A \alpha^{(0)}) \frac{\partial \alpha_k}{\partial \ln k^2} \right), \\
\frac{\partial \alpha_k}{\partial \ln k^2} &= \alpha_k \frac{\partial \ln Z_k^A}{\partial \ln k^2} + 2\left(\alpha_k - Z_k^A \alpha^{(0)} \right) \frac{\partial \ln g_k}{\partial \ln k^2} + \frac{3g_k^2}{(4\pi)^2} \left(h_A^\alpha(\mu_k, \alpha_k, Z_k^A \alpha^{(0)}) \frac{\partial \ln Z_k^A}{\partial \ln k^2} \right. \\
&\quad \left. + h_\mu^\alpha(\mu_k, \alpha_k, Z_k^A \alpha^{(0)}) \frac{\partial \mu_k}{\partial \ln k^2} + h_\alpha^\alpha(\mu_k, \alpha_k, Z_k^A \alpha^{(0)}) \frac{\partial \alpha_k}{\partial \ln k^2} \right).
\end{aligned}
\tag{37}
$$

The functions h_j^i, now depending additionally on $Z_k^A \alpha^{(0)}$, are given by the functions f_j^i appearing in (31), their derivatives with respect to μ_k and α_k, and some factors of μ_k, α_k and $Z_k^A \alpha^{(0)}$.

(36) and (37) together represent a system of five linear equations for the k–derivatives of the five parameters Z_k^A, Z_k^c, g_k, μ_k and α_k, where the coefficients are functions of g_k, μ_k, α_k and $Z_k^A \alpha^{(0)}$. Since only two of the latter parameters are independent by virtue of the modified STI, we conclude that the k–derivatives of all parameters are fixed by two independent quantities, say g_k and $Z_k^A \alpha^{(0)}$.

Choosing initial values for $g_{\bar{k}}$, $\alpha^{(0)}$ (and $Z_{\bar{k}}^A = 1$, $Z_{\bar{k}}^c = 1$), and determining the 'dependent' parameters $\mu_{\bar{k}}$ and $\alpha_{\bar{k}}$ with the help of the modified STI, we can now integrate the flow equations using (36) and (37). We remark that the division into dependent and independent parameters is not unique and that we could as well regard μ_k and α_k as independent parameters determining the values of g_k and $Z_k^A \alpha^{(0)}$.

Before we present the results of the integration of the flow equations let us, however, have a look at the way we can control the consistency of our approximation. In (37) we have taken $\partial \mu_k / \partial \ln k^2$ and $\partial \alpha_k / \partial \ln k^2$ from the k–derivative of the modified STI, but alternatively we could use flow equations derived from (16) for $\partial \mu_k / \partial \ln k^2$ and $\partial \alpha_k / \partial \ln k^2$ (as in the case of the other parameters). Due to the used truncations the corresponding results will generally be different, and this difference can be taken as a measure of the validity of the approximation.

Concerning the additional approximation of equal coupling constants, we could have taken the equation for $\partial \ln g_k / \partial \ln k^2$ in (36) just as well from the flow equation for the three–gluon or the Lcc–vertex. If the approximation $g_k^A = g_k^K = g_k^L$ is to hold, we should

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obtain at least approximately the same result for $\partial \ln g_k / \partial \ln k^2$ irrespective of the equation we use. This represents an additional condition to be fulfilled within the region of validity of our approximation.

Let us briefly discuss the Landau gauge $\alpha^{(0)} = 0$. In this case α_k is fixed to be zero for all values of k by the modified STI, as was remarked above. Hence the modified STI imply that $\alpha_k = 0$ is a fixed point in the Landau gauge. This is consistent with the flow equations, which demand that the k–derivative of α_k vanish at $\alpha_k = 0$.

We now turn to the solutions of the flow equations and start with the discussion of the one–loop approximation, i.e. we neglect terms of the order of g_k^4. Again, it is easy to solve (36) and (37) iteratively in the limit of small g_k, and we find

$$
\begin{aligned}
\frac{\partial \ln Z_k^A}{\partial \ln k^2} &= \frac{3g_k^2}{(4\pi)^2} \frac{13 - 3\alpha_k}{6} \\
\frac{\partial \ln Z_k^c}{\partial \ln k^2} &= \frac{3g_k^2}{(4\pi)^2} \frac{3 - \alpha_k}{4} \\
\frac{\partial \ln g_k}{\partial \ln k^2} &= -\frac{3g_k^2}{(4\pi)^2} \frac{11}{6} \\
\frac{\partial \mu_k}{\partial \ln k^2} &= 0 \\
\frac{\partial \alpha_k}{\partial \ln k^2} &= \alpha_k \frac{\partial \ln Z_k^A}{\partial \ln k^2} \, ,
\end{aligned}
\tag{38}
$$

where we have made use of the one–loop STI (33) in order to replace μ_k and $Z_k^A \alpha^{(0)}$ on the r.h.s. The first three equations represent the results familiar from perturbation theory, while the last two state that the modification of the STI does not affect the β–functions at the one–loop level (remember that $\alpha_k = Z_k^A \alpha^{(0)}$ in perturbation theory with BRST–invariant regularization).

Concerning the consistency of our approximation, we find that the flow equations for $\partial \mu_k / \partial \ln k^2$ and $\partial \alpha_k / \partial \ln k^2$ derived from (16) reproduce, to second order in g_k, exactly the results in (38). Here we have used (33) in order to rewrite the r.h.sides. Hence the flow equations are entirely consistent with the modified STI at the one–loop level. Likewise the expressions for $\partial \ln g_k / \partial \ln k^2$ taken from the flow equations for the three–gluon and the Lcc–vertex are identical with the one in (38) in second order in g_k. This is of course necessary for our approximation to be valid within the range of parameters where one expects the one–loop approximation to be reliable.

Let us now go beyond the one–loop level. Before we present the full solution, however, we would like to discuss briefly a different approximation that makes an analytical solution of the flow equations possible. It consists in arbitrarily setting μ_k equal to zero for all k, thereby ignoring the modified STI. Comparing with the full solution below where the modified STI are taken into account, we can thus estimate the effect of neglecting the gluon mass parameter. Furthermore we work in the Landau gauge.

Setting μ_k and α_k to zero, (36) becomes

$$
\begin{aligned}
\frac{\partial \ln Z_k^A}{\partial \ln k^2} &= \frac{3g_k^2}{(4\pi)^2}\left(\frac{13}{6} + \frac{1+150\ln(4/3)}{36}\cdot\frac{\partial \ln Z_k^A}{\partial \ln k^2} - \frac{1-36\ln(4/3)}{216}\cdot\frac{\partial \ln Z_k^c}{\partial \ln k^2}\right) \\
\frac{\partial \ln Z_k^c}{\partial \ln k^2} &= \frac{3g_k^2}{(4\pi)^2}\left(\frac{3}{4} + \frac{3\ln(4/3)}{4}\cdot\frac{\partial \ln Z_k^A}{\partial \ln k^2} + \frac{3\ln(4/3)}{4}\cdot\frac{\partial \ln Z_k^c}{\partial \ln k^2}\right) \\
\frac{\partial \ln g_k}{\partial \ln k^2} &= \frac{3g_k^2}{(4\pi)^2}\left(-\frac{11}{6} - \frac{1+204\ln(4/3)}{72}\cdot\frac{\partial \ln Z_k^A}{\partial \ln k^2} + \frac{1-360\ln(4/3)}{432}\cdot\frac{\partial \ln Z_k^c}{\partial \ln k^2}\right)
\end{aligned}
\tag{39}
$$

In second order in g_k we recover the one–loop results (38) with $\alpha_k = 0$.

From the solution of the system of linear equations (39) we obtain

$$
\frac{\partial \ln g_k}{\partial \ln k^2} = -\frac{2.6 \cdot 3g_k^2/(4\pi)^2 \cdot (2.7 - 3g_k^2/(4\pi)^2)}{(4.8 - 3g_k^2/(4\pi)^2)(0.81 - 3g_k^2/(4\pi)^2)} \,,
\tag{40}
$$

where we have given approximate numerical values for the appearing numbers.

Since SU(3)–Yang–Mills theory is asymptotically free, we start with a small coupling constant $g_{\bar{k}}$ at some large scale $\bar{k}$. Lowering the scale the coupling constant increases according to the solution of (40) until it reaches the point $3g_k^2/(4\pi)^2 = 0.81$, where the β–function given by (40) diverges. At this point k^2 as a function of g_k^2 reaches a maximum, which means that we cannot integrate the equation beyond this point. In contrast to perturbation theory, where we run into a Landau pole, the solution here just ends at a finite value of g_k^2.

In figure 3 we have plotted $\alpha_S = g_k^2/4\pi$ as a function of $\ln(\bar{k}^2/k^2)$ in this approximation (dot–dashed curve), together with the three–loop perturbative result (dotted curve) and the full solution to be discussed in the following.

Taking the modified STI into account, we determine the flow in the two–parameter space $((\alpha_k - 1)/(\alpha_k + 1), g_k^2/4\pi)$ (which maps values of α_k between zero and infinity into a finite interval). As discussed above, the k–derivatives of all parameters are already fixed by these two independent ones. One could have chosen another set as well, e.g. $Z_k^A \alpha^{(0)}$ instead of the above function of α_k, but the present choice turned out to be particularly convenient. The flow towards lower values of k in this parameter space induced by the flow equations is indicated in figure 4 by a vector field, where each vector represents the negative of the derivative with respect to $\ln k^2$. To make the directions and the magnitudes of all the vectors in the diagram visible, we had to rescale the vectors in a slightly unusual way, where the actual lengths L are replaced by $0.06 + 0.015 \cdot L$.

Starting with small values of g_k because of asymptotic freedom, we see from figure 4 that the flow is directed towards larger values of g_k, as expected from perturbation theory. The flow in the horizontal direction depends on the initial value of α_k. For $\alpha_k < 13/3$ the system is driven towards smaller values of α_k, for $\alpha_k > 13/3$ towards larger ones. The specific value $13/3$ can be read off from the one–loop formula (38). As shown in figure 4, this behaviour remains unchanged for larger values of g_k.

The curves appearing in the figure represent the boundary of the region where our approximation is valid. The different curves are obtained by imposing different constraints

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on the deviations resulting from our approximation as discussed above. The approximation can be considered valid below both curves and unreliable above. More details on the constraints determining the curves will be given below.

We find that for initial values $\alpha_k > 13/3$ the system flows out of the region of validity of the approximation quickly, the values of g_k still being small. Therefore we concentrate in the following on initial values $\alpha_k < 13/3$. Here the flow converges asymptotically towards $\alpha_k = 0$, the Landau gauge case. So $\alpha_k = 0$ is not only a fixed point of the flow, but even an attractive one. This is a very important feature because it shows how independence of the gauge parameter emerges in our formalism: Starting with an arbitrary value $\alpha_{\bar{k}} < 13/3$, corresponding to a certain value of $Z_{\bar{k}}^A \alpha^{(0)}$, the system flows towards $\alpha_k = 0$, and hence at a lower scale k the functions f_j^i and h_j^i appearing on the r.h.s. of the modified STI and the flow equations approach the form they have in the limit $\alpha_k \to 0$. Thus in the limit $k \to 0$ the parameters in the effective action become independent of $\alpha^{(0)}$, provided this feature persists beyond our reliability bounds.

Let us now discuss the curves in figure 4 representing the boundary of the region of validity of the approximation. From the above discussion it is clear that this region is determined by several conditions. First of all the deviations of the coupling constants g_k^A and g_k^L from g_k^K must be small. We can estimate the relative deviations from the last two equations in (31). Furthermore the k–derivatives of the parameters must not change drastically when we calculate them in one of the alternative ways described above. We concentrate here on the β–function of the coupling constant, i.e. on $\partial \ln g_k / \partial \ln k^2$. We require its relative deviation to be small when we use the flow equation for the three-gluon or the Lcc–vertex instead of the one for the $\bar{c}cA$–vertex in (36), and likewise when we take $\partial \mu_k / \partial \ln k^2$ or $\partial \alpha_k / \partial \ln k^2$ from (16) instead of using the k–derivative of the modified STI in (37). Now the solid curve in figure 4 represents the boundary of the region where all the relative deviations mentioned above are smaller than 3%, while for the region bounded by the dot–dashed curve they are smaller than 10%. Both curves consist of several smooth parts, each of which represents one specific constraint.

The region bounded by the dot–dashed curve extends to $g_k^2/4\pi \approx 2.8$ at $(\alpha_k - 1)/(\alpha_k + 1) \approx 0.2$. However to access this region starting with a small value $g_{\bar{k}}$ one would have to fine–tune the initial value $\alpha_{\bar{k}}$. The most interesting region due to the attractivity of the fixed point $\alpha_k = 0$ is the one with small values of α_k. Here the strongest constraints are obtained from replacing the equation for $\partial \ln g_k / \partial \ln k^2$ in (36) by the one taken from the flow equation for the three-gluon vertex for the solid curve and from replacing the equation for $\partial \mu_k / \partial \ln k^2$ in (37) by the one taken from (16) for the dot–dashed curve. Hence the point where our approximation is expected to become inappropriate is determined by the value of g_k where the flow equations and the k–derivative of the modified STI cease to be consistent. Figure 5 shows the β–function of g_k at $\alpha_k = 0$ as calculated from (36) and (37) (dot–dashed curve) and by taking $\partial \mu_k / \partial \ln k^2$ from (16) instead of using the k–derivative of the modified STI (solid curve). More precisely, $\partial \ln \alpha_S / \partial \ln k^2$ is plotted as a function of α_S, where $\alpha_S = g_k^2/4\pi$.

The result of the integrated flow equations is shown in figure 3 in the case of the Landau gauge $\alpha_k = 0$. Since $\alpha_k = 0$ is an attractive fixed point, the curves look the same for all

initial values $\alpha_{\bar{k}}$, provided the starting scale $\bar{k}$ is large enough. In figure 3 we have plotted $g_k^2/4\pi$ as a function of $\ln(\bar{k}^2/k^2)$, so the flow is directed towards the right. The initial value for the coupling constant is $g_{\bar{k}} = 1$. The figure shows, from left to right, the perturbative three–loop result [13], the full solution from the flow equations, and the analytical solution in the approximation $\mu_k = 0$. We see that the three curves are nearly identical as long as $\alpha_S = g_k^2/4\pi$ is smaller than 0.2 approximately, but already for $\alpha_S \approx 0.5$, where our approximation is still expected to be reliable, deviations become visible. We conclude that the effect of the non–vanishing gluon mass parameter becomes important here. Observe that the value of k^2 where our approximation ceases to be reliable, i.e. where $\alpha_S \approx 1.4$, lies beyond the Landau pole of three–loop perturbation theory.

Finally, let us have a look at the behaviour of the gluon mass parameter, again for $\alpha_k = 0$. Figure 6 shows the normalized mass parameter $m_k^2/\bar{k}^2$ as a function of $\ln(\bar{k}^2/k^2)$. It is driven to zero during the flow, and we emphasize again that this is nontrivial, since a slightly different starting point $m_{\bar{k}}^2$ is not expected to lead to a similar behaviour.

5   Conclusions and Outlook

To summarize, we have shown that the method proposed in [10] to use flow equations for the description of gauge theories is well suited for practical calculations. This has been demonstrated here for the case of a SU(3)–Yang–Mills theory in a simple but non–trivial approximation, where the k–dependent effective action is approximated by its perturbatively relevant part.

We have emphasized the crucial role played by the modified STI in this approach both for the general concept and for practical calculations, where approximations are inevitable. These identities allow to control the used approximation by providing internal consistency checks. We believe that this is a new and very useful property, which of course rests upon the local gauge symmetry of the original theory.

We have reproduced the usual perturbative one–loop results in the limit of small coupling constants, and we have studied in detail the behaviour and the effects of the gluon mass term (in particular on the higher order results), which is required by the modified STI. Finally we have argued how independence of the gauge fixing constant arises in our formalism.

The fact that the internal consistency checks did not allow us to go much beyond standard perturbation theory is an obvious concequence of our truncation of the effective action, not of the method per se: The discrepancies between flow equations and modified STI for small k tell us that higher dimensional operators are required in order to obtain a consistent description in this regime; this is not astonishing in view of the expected dynamical effects such as glueballs and condensates. Higher dimensional operators should allow, e.g., to describe an arbitrary momentum dependence of propagators and vertices. In the case of non-gauge theories, such more general parametrizations of effective actions have already been used for studies of phenomena like the formation of bound states, condensates, and other non–perturbative effects [5, 11]. The obvious next steps are now to investigate

these phenomena within non-abelian gauge theories, using generalisations of the method developed in the present paper.

Acknowledgements

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Figure captions

Figure 1: The dimensionless mass parameter $\mu_k = m_k^2/k^2$ as a function of the strong fine–structure constant $\alpha_S = g_k^2/4\pi$ for different values of the k–dependent gauge fixing constant α_k. The values depicted in the figure are $\alpha_k = 0$ (solid curve), 1 (dot–dashed curve) and 3 (dotted curve).

Figure 2: Flow equations for the one–particle irreducible n–point functions. The vertex functions are to be taken from $\hat{\Gamma}_k$, while the propagators are given in (29) and (30). The figure shows the flow equations for the gluon two–point function, the ghost two–point function and the $\bar{c}cA$–vertex.

Figure 3: The strong fine–structure constant $\alpha_S = g_k^2/4\pi$ as a function of the scale k in the Landau gauge $\alpha_k = 0$. The initial value at $k = \bar{k}$ is taken to be $g_{\bar{k}} = 1$. The figure depicts the perturbative three-loop result (dotted curve), the full solution from the flow equations (solid curve) and the solution in the approximation $\mu_k = 0$ (dot–dashed curve).

Figure 4: Representation of the flow in the two–parameter space $((\alpha_k - 1)/(\alpha_k + 1), \alpha_S = g_k^2/4\pi)$. The arrows are given by the negative of the derivatives with respect to $\ln k^2$, their lengths L rescaled to $0.06 + 0.015 \cdot L$ in the figure. The curves represent the boundary of the region of validity of the approximation. The solid curve results from the 3%–constraints and the dot–dashed curve from the 10%–constraints (see section 4 for details).

Figure 5: The β–function $\beta(\alpha_S) = \partial \ln \alpha_S / \partial \ln k^2$ as a function of the strong fine–structure constant $\alpha_S = g_k^2/4\pi$ in the Landau gauge $\alpha_k = 0$. The dot–dashed curve results from (36) and (37), while for the solid curve $\partial \mu_k / \partial \ln k^2$ is taken from (16). The deviation can be used to determine the region of validity of the approximation.

Figure 6: The normalized mass parameter $m_k^2/\bar{k}^2$ as a function of the scale k in the Landau gauge $\alpha_k = 0$. As in figure 3, the initial value at $k = \bar{k}$ is determined by $g_{\bar{k}} = 1$.
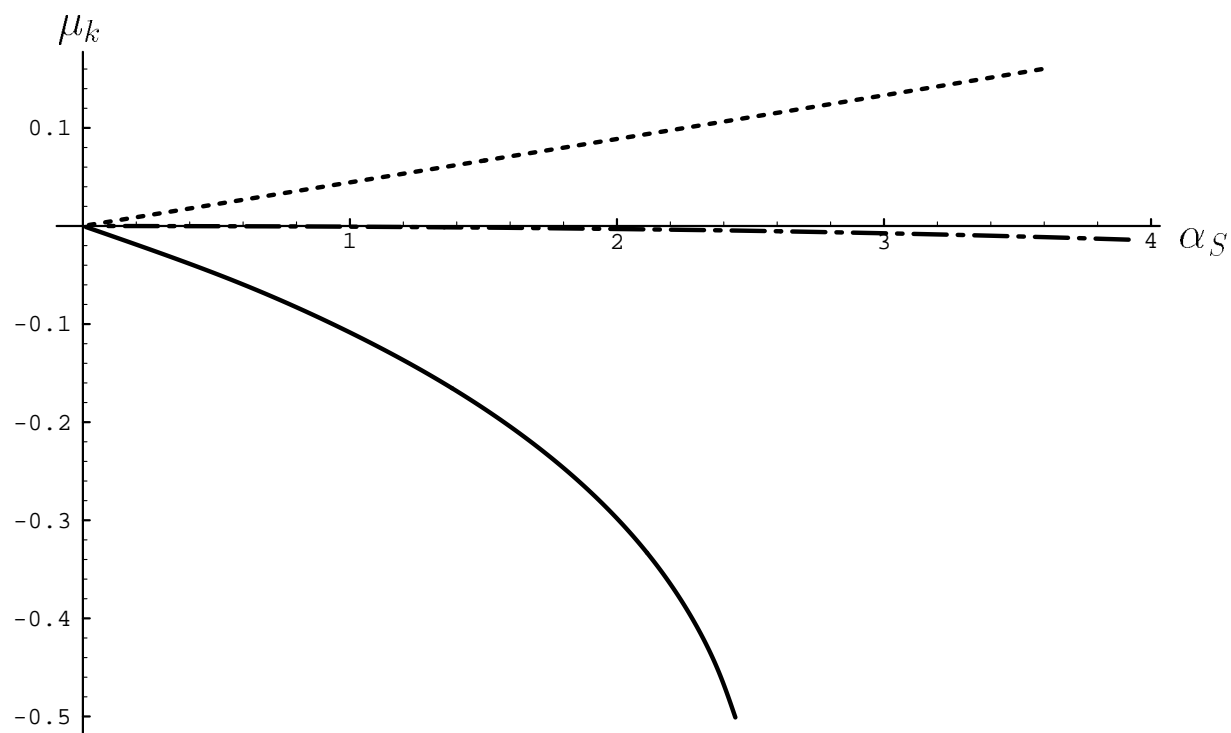
Figure 1
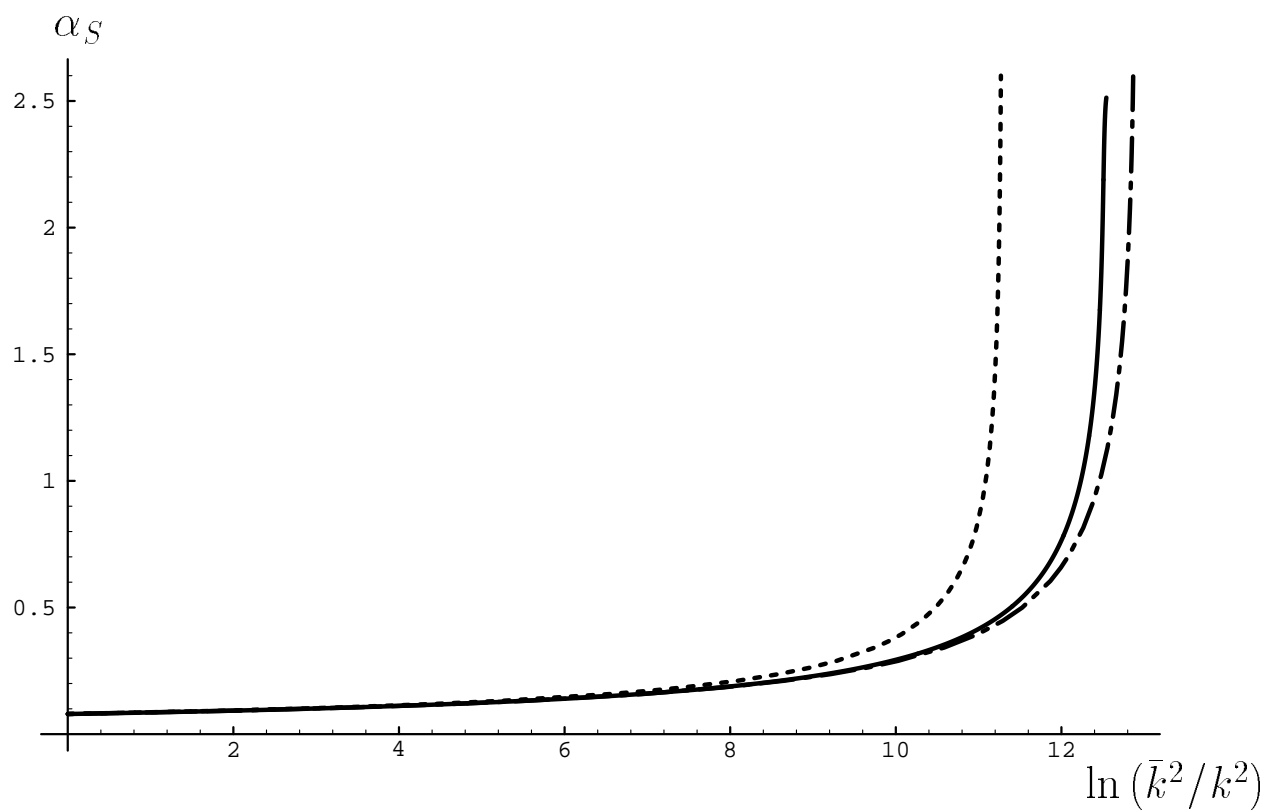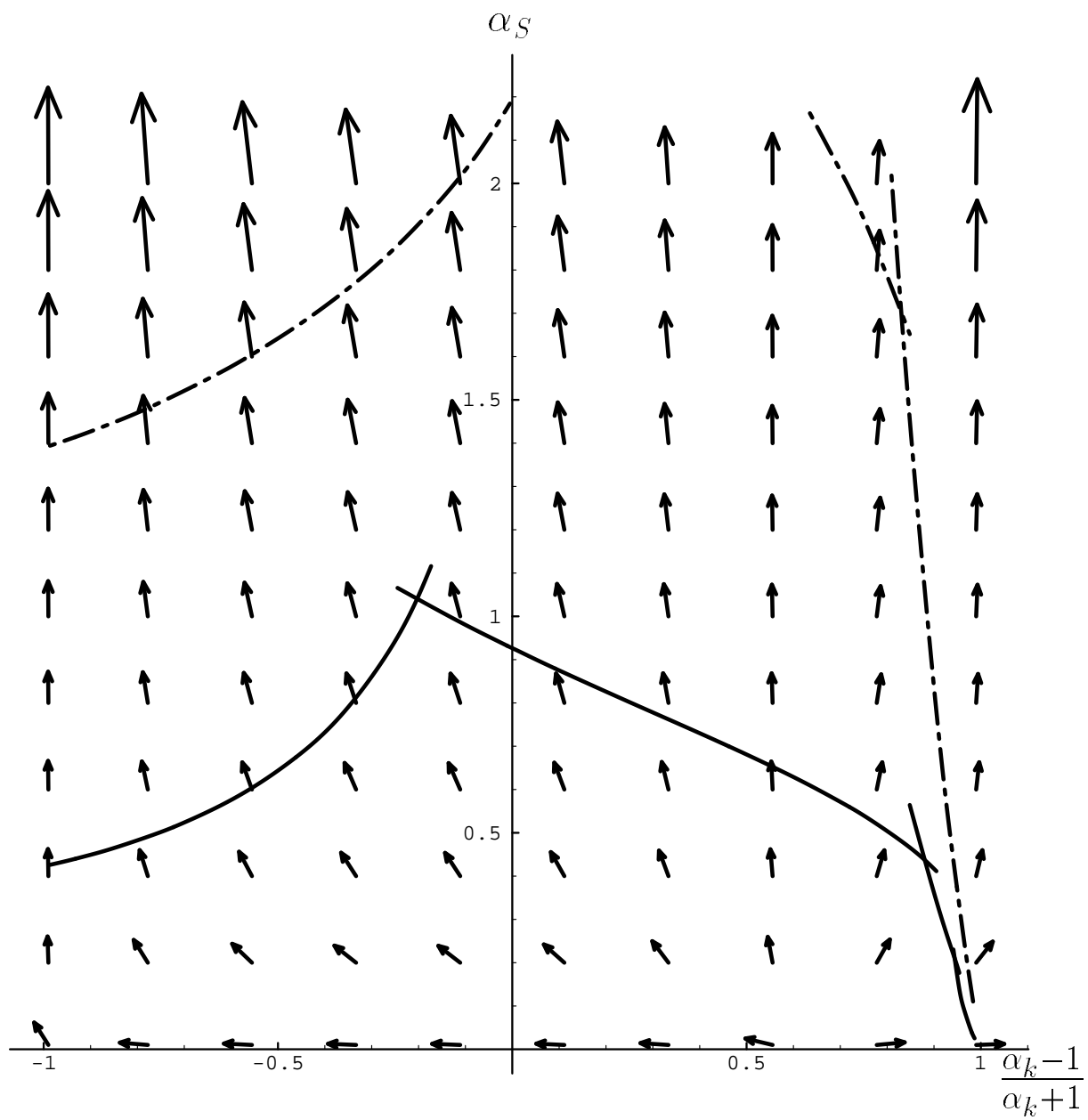
Figure 2

Figure 3

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